

# ADJOINT IDEALS ALONG CLOSED SUBVARIETIES OF HIGHER CODIMENSION

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**ABSTRACT.** In this paper, we introduce a notion of adjoint ideal sheaves along closed subvarieties of higher codimension and study its local properties using characteristic  $p$  methods. When  $X$  is a normal Gorenstein closed subvariety of a smooth complex variety  $A$ , we formulate a restriction property of the adjoint ideal sheaf  $\text{adj}_X(A)$  of  $A$  along  $X$  involving the l.c.i. ideal sheaf  $\mathcal{D}_X$  of  $X$ . The proof relies on a modification of generalized test ideals of Hara and Yoshida [11].

## INTRODUCTION

The adjoint ideal sheaf along a divisor  $D$  on a complex variety  $V$  is a modification of the multiplier ideal sheaf associated to  $D$ , and it encodes much information on the singularities of  $D$ . It recently turned out that it is a powerful tool in birational geometry and has several applications, such as the study of singularities of ample divisors of low degree on abelian varieties by Ein-Lazarsfeld [4] and Debarre-Hacon [3], inversion of adjunction on log canonicity proved by Kawakita [15], and the boundedness of pluricanonical maps of varieties of general type proved by Hacon-McKernan [8] and Takayama [23]. In this paper, we introduce a notion of adjoint ideal sheaves along closed subvarieties of higher codimension and study its local properties using characteristic  $p$  methods. We hope that our adjoint ideal sheaves lead to further applications.

Let  $A$  be a smooth complex variety and  $Y = \sum_{i=1}^m t_i Y_i$  be a formal combination, where the  $t_i$  are positive real numbers and the  $Y_i$  are proper closed subschemes of  $A$ . Let  $X$  be a reduced closed subscheme of pure codimension  $c$  of  $A$  such that no components of  $X$  are contained in the support of any  $Y_i$ . Suppose that  $\pi : \tilde{A} \rightarrow A$  is a log resolution of  $(A, X + Y)$  and  $E := \sum_{j=1}^s E_j$  is smooth, where  $E_1, \dots, E_s$  are all the irreducible divisors on  $\tilde{A}$  “dominating” a component of  $X$ . If  $K_{\tilde{A}/A}$  is the relative canonical divisor of  $\pi$ , then we define the *adjoint ideal sheaf*  $\text{adj}_X(A, Y)$  associated to the pair  $(A, Y)$  along  $X$  by

$$\text{adj}_X(A, Y) := \pi_* \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c \pi^{-1}(X) - \lfloor \pi^{-1}(Y) \rfloor + E),$$

where  $\pi^{-1}(X)$  and  $\pi^{-1}(Y) := \sum_{i=1}^m t_i \pi^{-1}(Y_i)$  are the scheme theoretic inverse images of  $X$  and  $Y$ , respectively (see Definition 1.6 for the precise definition of the adjoint ideal sheaf  $\text{adj}_X(A, Y)$ ). We say that  $(A, Y)$  is *purely log terminal* (plt, for short) along  $X$  if  $\text{adj}_X(A, Y) = \mathcal{O}_A$ . When  $X$  is a divisor, our definitions coincide with the definitions of usual plt pairs and adjoint ideal sheaves. In order to study local

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properties of our adjoint ideal sheaves using characteristic  $p$  methods, we consider a modification of generalized test ideals of Hara and Yoshida.

Let  $(R, \mathfrak{m})$  be a Noetherian local domain of characteristic  $p > 0$  and  $\underline{\mathfrak{a}}^t = \prod_{i=1}^m \mathfrak{a}_i^{t_i}$  be a formal combination, where the  $\mathfrak{a}_i \subseteq R$  are nonzero ideals and the  $t_i$  are positive real numbers. Hara-Yoshida [11] introduced notions of tight closure for the pair  $(R, \underline{\mathfrak{a}}^t)$ , called  $\underline{\mathfrak{a}}^t$ -tight closure, and the corresponding test ideal  $\tilde{\tau}(\underline{\mathfrak{a}}^t)$ . They then proved that the multiplier ideal sheaf coincides, after reduction to characteristic  $p \gg 0$ , with their generalized test ideal. In this paper, we define a notion of tight closure for a triple  $(R, I, \underline{\mathfrak{a}}^t)$ , called  $(I, \underline{\mathfrak{a}}^t)$ -tight closure, where  $I \subseteq R$  is an unmixed ideal of height  $c$  such that the  $\mathfrak{a}_i$  are not contained in any minimal prime ideal of  $I$ : the  $(I, \underline{\mathfrak{a}}^t)$ -tight closure  $J^{*(I, \underline{\mathfrak{a}}^t)}$  of an ideal  $J \subseteq R$  is the ideal consisting of all elements  $x \in R$  for which there exists  $\gamma \in R$  not in any minimal prime of  $I$  such that

$$\gamma I^{c(q-1)} \mathfrak{a}_1^{[t_1 q]} \cdots \mathfrak{a}_m^{[t_m q]} x^q \subseteq J^{[q]}$$

for all large  $q = p^e$ , where  $J^{[q]}$  is the ideal generated by the  $q^{\text{th}}$  powers of all elements of  $J$ . If  $N \subseteq M$  are  $R$ -modules, then the  $(I, \underline{\mathfrak{a}}^t)$ -tight closure  $N_M^{*(I, \underline{\mathfrak{a}}^t)}$  of  $N$  in  $M$  is defined similarly (see Definition 2.2 for the detail). We then define the generalized test ideal  $\tilde{\tau}_I(\underline{\mathfrak{a}}^t)$  along  $I$  to be the annihilator ideal of the  $(I, \underline{\mathfrak{a}}^t)$ -tight closure  $0_{E_R(R/\mathfrak{m})}^{*(I, \underline{\mathfrak{a}}^t)}$  of the zero submodule in the injective hull  $E_R(R/\mathfrak{m})$  of the residue field of  $R$ . When  $I = R$ ,  $(I, \underline{\mathfrak{a}}^t)$ -tight closure coincides with  $\underline{\mathfrak{a}}^t$ -tight closure and the generalized test ideal  $\tilde{\tau}_I(\underline{\mathfrak{a}}^t)$  along  $I$  is nothing but  $\tilde{\tau}(\underline{\mathfrak{a}}^t)$ . We conjecture that the ideal  $\tilde{\tau}_I(\underline{\mathfrak{a}}^t)$  corresponds to the adjoint ideal sheaf  $\text{adj}_X(A, Y)$ , and we obtain some partial results (Theorems 2.7 and 2.9). We use them to prove a restriction formula of our adjoint ideal sheaves.

Kawakita [16] and Ein-Musta a [5] introduced an ideal sheaf, called the l.c.i. defect ideal sheaf, which measures how far a variety is from being locally a complete intersection. They then proved a comparison of minimal log discrepancies of a variety  $X$  and its ambient space  $A$  with a boundary corresponding to the l.c.i. defect ideal sheaf  $\mathcal{D}_X$  of  $X$ . Their result inspires us to formulate a restriction property of the adjoint ideal sheaf  $\text{adj}_X(A, Y)$  involving the l.c.i. defect ideal sheaf  $\mathcal{D}_X$  of  $X$ .

**Theorem 3.1.** *Let  $A$  be a smooth complex variety and  $Y = \sum_{i=1}^m t_i Y_i$  be a formal combination, where the  $t_i$  are positive real numbers and the  $Y_i$  are proper closed subschemes of  $A$ . If  $X$  is a normal Gorenstein closed subvariety of codimension  $c$  of  $A$  which is not contained in the support of any  $Y_i$ , then*

$$\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X) = \text{adj}_X(A, Y) \mathcal{O}_X,$$

where  $\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X)$  is the multiplier ideal sheaf associated to the pair  $(X, V(\mathcal{D}_X) + Y|_X)$  (see Definition 1.2 for the definition of multiplier ideal sheaves).

By making use of the partial correspondence between the adjoint ideal sheaf  $\text{adj}_X(A, Y)$  and the generalized test ideal  $\tilde{\tau}_I(\underline{\mathfrak{a}}^t)$ , Theorem 3.1 can be reduced to a purely algebraic problem on some ideals of a ring of characteristic  $p > 0$ . We then solve the problem using the linkage theory of Peskine and Szpiro [19].

As a corollary of Theorem 3.1, we obtain a characterization of being plt along a Gorenstein closed subvariety in terms of Frobenius splitting (Corollary 3.4).

## 1. MULTIPLIER IDEALS AND ADJOINT IDEALS

In this section, we first recall the definitions of multiplier ideal sheaves and adjoint ideal sheaves along divisors (our main references are [17] and [18]), and then we introduce a notion of adjoint ideal sheaves along closed subvarieties of higher codimension.

Let  $X$  be a  $d$ -dimensional  $\mathbb{Q}$ -Gorenstein normal variety over a field  $k$  of characteristic zero and  $Y = \sum_{i=1}^m t_i Y_i$  be a formal combination, where the  $t_i$  are real numbers and the  $Y_i$  are proper closed subschemes of  $X$ . Since  $X$  is normal, we have a Weil divisor  $K_X$  on  $X$ , uniquely determined up to linear equivalence, such that  $\mathcal{O}_X(K_X) \cong i_* \Omega_{X_{\text{reg}}}^d$  where  $i : X_{\text{reg}} \hookrightarrow X$  is the inclusion of the nonsingular locus. Moreover, since  $X$  is  $\mathbb{Q}$ -Gorenstein, there exists a positive integer  $r$  such that  $rK_X$  is a Cartier divisor.

Let  $E$  be a *divisor over*  $X$ , that is,  $E$  is an irreducible divisor on some normal variety  $X'$  with a birational morphism  $f : X' \rightarrow X$ . We identify two divisors over  $X$  if they correspond to the same valuation of the function field  $k(X)$ . The *center* of  $E$  is the closure of  $f(E)$  in  $X$ , denoted by  $c_X(E)$ . If  $Z$  is a closed subscheme of  $X$ , then we define  $\text{ord}_E(Z)$  as follows: we may assume that the scheme theoretic inverse image  $f^{-1}(Z)$  is a divisor. Then  $\text{ord}_E(Z)$  is the coefficient of  $E$  in  $f^{-1}(Z)$ . We put  $\text{ord}_E(Y) := \sum_{i=1}^m t_i \text{ord}_E(Y_i)$  and define  $\text{ord}_E(K_{-/X})$  as the coefficient of  $E$  in the relative canonical divisor  $K_{X'/X}$  of  $f$ . Recall that  $K_{X'/X}$  is the unique  $\mathbb{Q}$ -divisor supported on the exceptional locus of  $f$  such that  $rK_{X'/X}$  is linearly equivalent to  $rK_{X'} - f^*(rK_X)$ . Then the *log discrepancy*  $a(E; X, Y)$  of  $(X, Y)$  with respect to  $E$  is

$$a(E; X, Y) := \text{ord}_E(K_{-/X}) - \text{ord}_E(Y) + 1.$$

If  $W$  is a closed subset of  $X$ , then the *minimal log discrepancy*  $\text{mld}(W; X, Y)$  of  $(X, Y)$  along  $W$  is defined by

$$\text{mld}(W; X, Y) := \inf\{a(E; X, Y) \mid E \text{ is a divisor over } X, c_X(E) \subseteq W\}.$$

**Definition 1.1.** Let the notation be the same as above.

- (i) We say that the pair  $(X, Y)$  is *Kawamata log terminal* (klt, for short) if  $\text{mld}(X; X, Y) > 0$ . Since a resolution of singularities is obtained by blowing up subvarieties in the singular locus, this condition is equivalent to saying that  $\text{mld}(X_{\text{sing}} \cup \bigcup_{i=1}^m Y_i; X, Y) > 0$ , where  $X_{\text{sing}}$  is the singular locus of  $X$ .
- (ii) Let  $D$  be a reduced Cartier divisor on  $X$  such that no components of  $D$  are contained in the support of any  $Y_i$ . Then we say that  $(X, Y)$  is *purely log terminal* (plt, for short) along  $D$  if  $a(E; X, D + Y) > 0$  for all divisors  $E$  over  $X$  dominating no components of  $D$ .

Suppose that  $(X, Y)$  is a pair as above. A *log resolution* of the pair  $(X, Y)$  is a proper birational morphism  $\pi : \tilde{X} \rightarrow X$  with  $\tilde{X}$  nonsingular such that all the scheme theoretic inverse images  $\pi^{-1}(Y_i)$  are divisors and in addition  $\bigcup_{i=1}^m \text{Supp } \pi^{-1}(Y_i) \cup \text{Exc}(\pi)$  is a simple normal crossing divisor. The existence of log resolutions is guaranteed by Hironaka's desingularization theorem [13].

**Definition 1.2** ([18, Definition 9.3.60]). Let the notation be the same as above.

- (i) Fix a log resolution  $\pi : \tilde{X} \rightarrow X$  of  $(X, Y)$ . The *multiplier ideal sheaf*  $\mathcal{J}(X, Y)$  associated to the pair  $(X, Y)$  is

$$\mathcal{J}(X, Y) = \pi_* \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - \sum_{i=1}^m t_i \pi^{-1}(Y_i) \rceil) \subseteq \mathcal{O}_X.$$

- (ii) Let  $D$  be a reduced Cartier divisor on  $X$  such that no components of  $D$  are contained in the support of any  $Y_i$ . Fix a log resolution  $\pi : \tilde{X} \rightarrow X$  of  $(X, D + Y)$  so that the strict transform  $\pi_*^{-1}D$  of  $D$  is nonsingular (but possibly disconnected). Then the *adjoint ideal sheaf*  $\text{adj}_D(X, Y)$  associated to the pair  $(X, Y)$  along  $D$  is

$$\text{adj}_D(X, Y) = \pi_* \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - \sum_{i=1}^m t_i \pi^{-1}(Y_i) - \pi^*D + \pi_*^{-1}D \rceil) \subseteq \mathcal{O}_X.$$

We denote this ideal sheaf simply by  $\text{adj}_D(X)$  when  $Y = 0$ .

*Remark 1.3.* (1) (cf. [18, Theorem 9.2.18])  $\mathcal{J}(X, Y)$  and  $\text{adj}(X, Y)$  are independent of the choice of the log resolution  $\pi$  used to define them (see also Lemma 1.7 (1)).

(2) The pair  $(X, Y)$  is klt (resp. plt along  $D$ ) if and only if  $\mathcal{J}(X, Y) = \mathcal{O}_X$  (resp.  $\text{adj}_D(X, Y) = \mathcal{O}_X$ ).

(3) ([18, Example 9.3.49]) Suppose that  $X$  is an affine variety and  $I$  is a nonzero ideal of  $\mathcal{O}_X$ . Choose a general element  $f$  in  $I$  so that  $\text{div}_X(f)$  is reduced and no components of  $\text{div}_X(f)$  are contained in the support of any  $Y_i$ . Then

$$\mathcal{J}(X, V(I) + Y) = \text{adj}_{\text{div}_X(f)}(X, Y)$$

(see also Claim 1 in the proof of Theorem 3.1).

An analogue of local vanishing theorem [18, Theorem 9.4.1] holds for the adjoint ideal sheaf  $\text{adj}_D(X, Y)$  along a divisor  $D$ .

**Proposition 1.4.** *Let the notation be the same as in Definition 1.2 (ii). Then for all  $i > 0$ ,*

$$R^i \pi_* \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - \pi^{-1}(Y) - \pi^*D + \pi_*^{-1}D \rceil) = 0.$$

*Proof.* Set  $B := \lceil K_{\tilde{X}/X} - \pi^{-1}(Y) - \pi^*D \rceil$  and  $\tilde{D} := \pi_*^{-1}D$ . Let  $\nu : D^\nu \rightarrow D$  be the normalization of  $D$ ,  $\mu : \tilde{D} \rightarrow D^\nu$  be the induced morphism and  $\pi_D : \tilde{D} \rightarrow D$  be the composite morphism. Then there exists an effective  $\mathbb{Q}$ -divisor  $\text{Diff}_{D^\nu}(0)$  on  $D^\nu$ , called the *different* of the zero divisor on  $D^\nu$  (see [20, §3] for details), such that  $K_{D^\nu} + \text{Diff}_{D^\nu}(0)$  is  $\mathbb{Q}$ -Cartier and  $K_{D^\nu} + \text{Diff}_{D^\nu}(0) = \nu^*((K_X + D)|_D)$ . Now we have the following exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(B) \rightarrow \mathcal{O}_{\tilde{X}}(B + \tilde{D}) \rightarrow \mathcal{O}_{\tilde{D}}(\lceil K_{\tilde{D}} - \mu^*(K_{D^\nu} + \text{Diff}_{D^\nu}(0)) - \pi_D^{-1}(Y|_D) \rceil) \rightarrow 0.$$

It follows from Kawamata-Viehweg vanishing theorem that

$$R^i \pi_* \mathcal{O}_{\tilde{X}}(B) = R^i \pi_{D*} \mathcal{O}_{\tilde{D}}(\lceil K_{\tilde{D}} - \mu^*(K_{D^\nu} + \text{Diff}_{D^\nu}(0)) - \pi_D^{-1}(Y|_D) \rceil) = 0$$

for all  $i > 0$ . Thus, we have  $R^i \pi_* \mathcal{O}_{\tilde{X}}(B + \tilde{D}) = 0$  for all  $i > 0$ .  $\square$

**Example 1.5.** Let  $X = \mathbb{C}^2 = \text{Spec } \mathbb{C}[x, y]$  be the two-dimensional affine space and let  $D = (x^3 + y^5 = 0) \subseteq X$ . Then  $\text{adj}_D(X) = (x^2, xy, y^3)$ , whereas  $\mathcal{J}(X, D) = (x^3 + y^5)$ .

When the ambient variety is smooth, we can generalize the notion of adjoint ideal sheaves to the higher codimension case.

Let  $A$  be a nonsingular variety over a field  $k$  of characteristic zero and  $Y = \sum_{i=1}^m t_i Y_i$  be a formal combination, where the  $t_i$  are positive real numbers and the  $Y_i$  are proper closed subschemes of  $A$ . Let  $X$  be a reduced closed subscheme of pure codimension  $c$  of  $A$  such that no components of  $X$  are contained in the support of any  $Y_i$ . Let  $f : A' := \text{Bl}_X A \rightarrow A$  be the blowing-up of  $A$  along  $X$  and  $E_1, \dots, E_s$  be all the components of the exceptional divisor of  $f$  dominating an irreducible component of  $X$ . Fix a log resolution  $g : \tilde{A} \rightarrow A'$  of  $(A', f^{-1}(X) + f^{-1}(Y))$  such that  $\sum_{j=1}^s g_*^{-1} E_j$  is nonsingular (but possibly disconnected), and put  $\pi := f \circ g : \tilde{A} \rightarrow A$ .

**Definition 1.6.** In the above situation, the *adjoint ideal sheaf*  $\text{adj}_X(A, Y)$  associated to the pair  $(A, Y)$  along  $X$  is

$$\text{adj}_X(A, Y) = \pi_* \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - \sum_{i=1}^m \lfloor t_i \pi^{-1}(Y_i) \rfloor - c \pi^{-1}(X) + \sum_{j=1}^s g_*^{-1} E_j) \subseteq \mathcal{O}_A.$$

We denote this ideal sheaf simply by  $\text{adj}_X(A)$  when  $Y = 0$ . We say that  $(A, Y)$  (resp.  $A$ ) is *purely log terminal* (plt, for short) along  $X$  if  $\text{adj}_X(A, Y) = \mathcal{O}_A$  (resp.  $\text{adj}_X(A) = \mathcal{O}_A$ ). When  $X$  is a divisor, these definitions coincide with those given in Definition 1.1 (ii) and Definition 1.2 (ii).

**Lemma 1.7.** *Let the notation be as in Definition 1.6.*

- (1) *The adjoint ideal sheaf  $\text{adj}_X(A, Y)$  is independent of the choice of the log resolution used to define it.*
- (2)  *$(A, Y)$  is plt along  $X$  if and only if*

$$\text{mld}(X_{\text{sing}} \cup \bigcup_{i=1}^m Y_i; A, cX + Y) > 0,$$

*where  $X_{\text{sing}}$  is the singular locus of  $X$ . More generally, the adjoint ideal sheaf  $\text{adj}_X(A, Y)$  is an ideal sheaf of  $X$  whose sections over an open subset  $U$  are those  $\varphi \in \mathcal{O}_X(U)$  such that for every divisor  $E$  over  $X$  whose center intersects  $U$  and is contained in  $X_{\text{sing}} \cup \bigcup_{i=1}^m Y_i$ ,*

$$\text{ord}_E(\varphi) + a(E; A, cX + Y) > 0.$$

*Proof.* Let  $f : A' \rightarrow A$  be the blowing-up of  $A$  along  $X$  and  $E_1, \dots, E_s$  be all the components of the exceptional divisor of  $f$  dominating an irreducible component of  $X$ . Put  $E = E_1 + \dots + E_s$ .

(1) The proof is essentially the same as that of [18, Theorem 9.2.18]. We consider a sequence of morphisms  $V \xrightarrow{\nu} \tilde{A} \xrightarrow{\pi} A$ , where  $\pi$  is a log resolution of  $(A, X + Y)$  such that the strict transform  $\tilde{E}$  of  $E$  is nonsingular and  $\nu$  is a log resolution of  $(\tilde{A}, \pi^{-1}(X) + \pi^{-1}(Y))$ .

*Claim.* Let  $D$  be a reduced disconnected divisor on  $\tilde{A}$  with simple normal crossing support and  $B$  be an  $\mathbb{R}$ -divisor on  $\tilde{A}$  with simple normal crossing support which has no common components with  $D$ . Suppose that  $\mu : W \rightarrow \tilde{A}$  is a log resolution of  $D + B$ . Then

$$\mu_* \mathcal{O}_W(K_{W/\tilde{A}} - \lfloor \mu^*(D + B) \rfloor + \mu_*^{-1} D) = \mathcal{O}_{\tilde{A}}(-\lfloor B \rfloor).$$

*Proof of Claim.* It follows from the projection formula that if the assertion holds for a given  $\mathbb{R}$ -divisor  $B$ , then it holds also for  $B + B'$  whenever  $B'$  is an integral divisor on  $\tilde{A}$ . Therefore, we may assume that  $\lfloor B \rfloor = 0$ . Setting  $\Delta = D + B$ , we have

$$K_W + \mu_*^{-1} \Delta = \mu^*(K_{\tilde{A}} + \Delta) + \sum_F (a(F; \tilde{A}, \Delta) - 1) F,$$

where  $F$  runs through all  $\mu$ -exceptional prime divisors on  $W$ . Then

$$K_{W/\tilde{A}} - \lfloor \mu^* \Delta \rfloor + \mu_*^{-1} D = \lceil K_{W/\tilde{A}} - \mu^* \Delta + \mu_*^{-1} \Delta \rceil = \sum_F \lceil a(F; \tilde{A}, \Delta) - 1 \rceil F,$$

because  $\lfloor B \rfloor = 0$ . Since  $D$  is disconnected, by [17, Corollary 2.31 (3)], one has  $a(F; \tilde{A}, \Delta) > 0$  for all  $\mu$ -exceptional prime divisors  $F$ . This completes the proof.  $\square$

By the above claim, we know that

$$\nu_* \mathcal{O}_V(K_{V/\tilde{A}} - \lfloor \nu^{-1} \pi^{-1}(Y) \rfloor - \nu^* \tilde{E} + \nu_*^{-1} \tilde{E}) = \mathcal{O}_{\tilde{A}}(-\lfloor \pi^{-1}(Y) \rfloor).$$

Then, setting  $h := \pi \circ \nu$ , one finds using the projection formula:

$$\begin{aligned} & h_* \mathcal{O}_V(K_{V/\tilde{A}} - \lfloor h^{-1}(Y) \rfloor - c h^{-1}(X) + \nu_*^{-1} \tilde{E}) \\ &= \pi_* \nu_* \left( \nu^* \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c \pi^{-1}(X) + \tilde{E}) \otimes \mathcal{O}_V(K_{V/\tilde{A}} - \lfloor \nu^{-1} \pi^{-1}(Y) \rfloor - \nu^* \tilde{E} + \nu_*^{-1} \tilde{E}) \right) \\ &= \pi_* \left( \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c \pi^{-1}(X) + \tilde{E}) \otimes \nu_* \mathcal{O}_V(K_{V/\tilde{A}} - \lfloor \nu^{-1} \pi^{-1}(Y) \rfloor - \nu^* \tilde{E} + \nu_*^{-1} \tilde{E}) \right) \\ &= \pi_* \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - \lfloor \pi^{-1}(Y) \rfloor - c \pi^{-1}(X) + \tilde{E}). \end{aligned}$$

In other words, we obtain the same adjoint ideal sheaf working from  $h$  as working from  $\pi$ . Since any two resolutions can be dominated by a third, the assertion follows.

(2) Since no components of  $X$  are contained in the support of any  $Y_i$ ,  $Y$  does not contribute to  $a(E_i; A, cX + Y)$ . By [17, Lemma 2.29], one has  $a(E_i; A, cX + Y) = a(E_i; A, cX) = 0$  for all  $i = 1, \dots, s$ . We have already seen in (1) that the adjoint ideal sheaf is independent of the choice of the log resolution used to define it. Since

$$\begin{aligned} K_{\tilde{A}/A} - \lfloor \pi^{-1}(Y) \rfloor - c \pi^{-1}(X) + g_*^{-1} E &= \sum_{F: \text{divisor on } \tilde{A}} \lceil a(F; A, cX + Y) - 1 \rceil F + g_*^{-1} E \\ &= \sum_{F \neq g_*^{-1} E_i} \lceil a(F; A, cX + Y) - 1 \rceil F \end{aligned}$$

for every log resolution  $g : \tilde{A} \rightarrow A'$  of  $(A', f^{-1}(X) + f^{-1}(Y))$  where  $\pi = f \circ g : \tilde{A} \rightarrow A$ , the pair  $(A, Y)$  is plt along  $X$  if and only if  $a(F; A, cX + Y) > 0$  for every divisor  $F$  over  $A$  dominating no components of  $E$ . This implies that if  $(A, Y)$  is plt along  $X$ , then  $\text{mld}(X_{\text{sing}} \cup \bigcup_{i=1}^m Y_i; A, cX + Y) > 0$ . The converse follows from the fact that

a log resolution of  $(A', f^{-1}(X) + f^{-1}(Y))$  is obtained by blowing up subvarieties in  $f^{-1}(X_{\text{sing}} \cup \bigcup_{i=1}^n Y_i)$ . We can prove the general case similarly.  $\square$

**Example 1.8.** (1) Suppose that  $X$  is locally a complete intersection variety, and consider the blowing-up  $f : A' = \text{Bl}_X A \rightarrow A$  of  $A$  along  $X$ . Then the exceptional divisor  $E := f^{-1}(X)$  is a projective bundle over  $X$ . In particular,  $E$  is normal and locally a complete intersection, and therefore, so is  $A'$ . By Hironaka's embedded resolution of singularities [13], there exists a log resolution  $g : \tilde{A} \rightarrow A'$  of  $(A', E)$  which is an isomorphism over the complement of a proper closed subset of  $E$ . Let  $\tilde{E}$  be the strict transform of  $E$  on  $\tilde{A}$ , and put  $\pi := f \circ g : \tilde{A} \rightarrow A$ . Since

$$K_{\tilde{A}/A} - c \pi^{-1}(X) + \tilde{E} = K_{\tilde{A}/A'} - g^*E + \tilde{E},$$

$A$  is plt along  $X$  if and only if  $A'$  is plt along  $E$ . Since  $A' \setminus E$  is nonsingular, a result of Kollár [17, Theorem 5.50] says that  $A'$  is plt along  $E$  if and only if  $E$  is klt. This means that  $X$  is klt, because  $E$  is locally a product of  $X$  and an affine space. It therefore follows that  $A$  is plt along  $X$  if and only if  $X$  is klt. That is,  $\text{adj}_X(A)$  defines the non-klt locus of  $X$ .

(2) Let  $X = \frac{1}{3}(1, 1, 1)$  be the quotient of  $\mathbb{C}^3 = \text{Spec } \mathbb{C}[x_1, x_2, x_3]$  by the action of  $\mathbb{Z}/3\mathbb{Z}$  given by  $x_i \mapsto \xi x_i$ , where  $\xi$  is a primitive cubic root of unity.  $X$  can be embedded into  $A := \mathbb{C}^{10}$ , and we will compute the ideal sheaf  $\text{adj}_X(A)$ . Let  $\pi_1 : A_1 \rightarrow A$  be the blowing-up of  $A$  at the origin with exceptional divisor  $E_1$  (we use the same letter for its strict transform). Then the weak transform  $X_1$  of  $X$  is nonsingular. Next, let  $\pi_2 : A_2 \rightarrow A_1$  be the blowing-up of  $A_1$  along  $X_1$  with exceptional divisor  $E_2$ . Setting  $\pi := \pi_1 \circ \pi_2 : A_2 \rightarrow A$ , we have  $K_{A_2/A} = K_{A_2/A_1} + \pi_2^* K_{A_1/A} = 9E_1 + 6E_2$  and  $\pi^{-1}(X) = 2E_1 + E_2$ . Thus,

$$\text{adj}_X(A) = \pi_* \mathcal{O}_{A_2}(K_{A_2/A} - 7\pi^{-1}(X) + E_2) = \pi_* \mathcal{O}_{A_2}(-5E_1) = \mathfrak{m}_{A,0}^5,$$

where  $\mathfrak{m}_{A,0} \subseteq \mathcal{O}_A$  is the maximal ideal sheaf of the origin.

(3) Let  $X$  be the quotient of  $(x^2 + y^3 + z^6 = 0) \subset \mathbb{C}^3$  by the action of  $\mathbb{Z}/5\mathbb{Z}$  given by  $x \mapsto \xi^3 x$ ,  $y \mapsto \xi^2 y$  and  $z \mapsto \xi z$ , where  $\xi$  is a primitive quintic root of unity. Then  $X$  can be embedded into  $A := \mathbb{C}^5$ , and by an argument similar to that of (2), we have  $\text{adj}_X(A) = \mathfrak{m}_{A,0}^2$ , where  $\mathfrak{m}_{A,0} \subseteq \mathcal{O}_A$  is the maximal ideal sheaf of the origin.

## 2. A MODIFICATION OF GENERALIZED TEST IDEALS

In this section, we consider a modification of generalized test ideals of Hara and Yoshida [11], which conjecturally corresponds to our adjoint ideal sheaf.

Throughout this paper, all rings are Noetherian commutative rings with identity. For an integral domain  $R$  and an unmixed ideal  $I$  of  $R$ , we denote by  $R^{\circ, I}$  the set of elements of  $R$  that are not in any minimal prime ideal of  $I$ .

Let  $R$  be an integral domain of characteristic  $p > 0$ . For an ideal  $J$  of  $R$  and a power  $q$  of  $p$ , we denote by  $J^{[q]}$  the ideal of  $R$  generated by the  $q^{\text{th}}$  powers of all elements of  $J$ . Let  $F : R \rightarrow R$  be the Frobenius map, that is, the ring homomorphism sending  $x$  to  $x^p$ . The ring  $R$  viewed as an  $R$ -module via the  $e$ -times iterated Frobenius map  $F^e : R \rightarrow R$  is denoted by  ${}^e R$ . Since  $R$  is reduced,  $F^e : R \rightarrow {}^e R$  is identified with the natural inclusion map  $R \hookrightarrow R^{1/p^e}$ . We say that  $R$  is  $F$ -finite if

${}^1R$  (or  $R^{1/p}$ ) is a finitely generated  $R$ -module. For example, any algebra essentially of finite type over a perfect field is F-finite.

**Definition 2.1** ([Ta1, Definition 3.1]). Let  $R$  be an F-finite domain of characteristic  $p > 0$  and  $\underline{\mathfrak{a}}^t = \prod_{i=1}^m \mathfrak{a}_i^{t_i}$  be a formal combination, where the  $\mathfrak{a}_i$  are nonzero ideals of  $R$  and the  $t_i$  are positive real numbers.

- (i) The pair  $(R, \underline{\mathfrak{a}}^t)$  is said to be *strongly F-regular* if for every  $\gamma \in R^\circ$ , there exist  $q = p^e$  and  $\delta \in \mathfrak{a}_1^{[t_1 q]} \cdots \mathfrak{a}_m^{[t_m q]}$  such that  $(\gamma\delta)^{1/q}R \hookrightarrow R^{1/q}$  splits as an  $R$ -module homomorphism.
- (ii) Let  $I \subseteq R$  be an unmixed ideal of height  $c$  and suppose that  $\mathfrak{a}_i \cap R^{\circ, I} \neq \emptyset$  for all  $i = 1, \dots, m$ . Then the pair  $(R, \underline{\mathfrak{a}}^t)$  is said to be *purely F-regular* along  $I$  if for every  $\gamma \in R^{\circ, I}$  there exist  $q = p^e$  and  $\delta \in I^{c(q-1)} \mathfrak{a}_1^{[t_1 q]} \cdots \mathfrak{a}_m^{[t_m q]}$  such that  $(\gamma\delta)^{1/q}R \hookrightarrow R^{1/q}$  splits as an  $R$ -module homomorphism. We say that  $R$  is purely F-regular along  $I$  if so is the pair  $(R, R^1)$ .

Let  $R$  be an integral domain of characteristic  $p > 0$  and  $M$  be an  $R$ -module. For each  $q = p^e$ , we denote  $\mathbb{F}^e(M) = \mathbb{F}_R^e(M) := {}^eR \otimes_R M$  and regard it as an  $R$ -module by the action of  $R = {}^eR$  from the left. Then we have the  $e$ -times iterated Frobenius map  $F_M^e: M \rightarrow \mathbb{F}^e(M)$  induced on  $M$ . The image of an element  $z \in M$  via this map is denoted by  $z^q := F_M^e(z) \in \mathbb{F}^e(M)$ . For an  $R$ -submodule  $N$  of  $M$ , we denote by  $N_M^{[q]}$  the image of the induced map  $\mathbb{F}^e(N) \rightarrow \mathbb{F}^e(M)$ . If  $I$  is an ideal of  $R$ , then  $I_R^{[q]} = I^{[q]}$ .

Now we introduce a new generalization of tight closure and the corresponding test ideal.

**Definition 2.2.** Let  $R$  be an excellent domain of characteristic  $p > 0$  and  $I \subseteq R$  be an unmixed ideal of height  $c$ . Let  $\underline{\mathfrak{a}}^t = \prod_{i=1}^m \mathfrak{a}_i^{t_i}$  be a formal combination, where the  $\mathfrak{a}_i$  are ideals of  $R$  such that  $\mathfrak{a}_i \cap R^{\circ, I} \neq \emptyset$  and the  $t_i$  are positive real numbers.

- (i) If  $N \subseteq M$  are  $R$ -modules, then the  $(I, \underline{\mathfrak{a}}^t)$ -tight closure  $N_M^{*(I, \underline{\mathfrak{a}}^t)}$  of  $N$  in  $M$  is defined to be the submodule of  $M$  consisting of all elements  $z \in M$  for which there exists  $\gamma \in R^{\circ, I}$  such that

$$\gamma I^{c(q-1)} \mathfrak{a}_1^{[t_1 q]} \cdots \mathfrak{a}_m^{[t_m q]} z^q \subseteq N_M^{[q]}$$

for all large  $q = p^e$ .

- (ii) Let  $E = \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$  be the direct sum, taken over all maximal ideals  $\mathfrak{m}$  of  $R$ , of the injective hulls of the residue fields  $R/\mathfrak{m}$ . The *generalized test ideal*  $\tilde{\tau}_I(R, \underline{\mathfrak{a}}^t)$  associated to the pair  $(R, \underline{\mathfrak{a}}^t)$  along  $I$  is

$$\tilde{\tau}_I(R, \underline{\mathfrak{a}}^t) = \text{Ann}_R(0_E^{*(I, \underline{\mathfrak{a}}^t)}) \subseteq R.$$

We denote this ideal simply by  $\tilde{\tau}_I(R)$  when  $\mathfrak{a}_i = R$  for all  $i = 1, \dots, m$ .

*Remark 2.3.* When  $R$  is a normal domain and  $I = xR$  is a principal ideal,  $(I, \underline{\mathfrak{a}}^t)$ -tight closure coincides with divisorial  $(\text{div}(x), \underline{\mathfrak{a}}^t)$ -tight closure introduced in [22].

**Definition-Lemma 2.4.** Let  $R$  be an excellent domain of characteristic  $p > 0$  and  $I$  be an unmixed ideal of height  $c$ . Let  $E = \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$  be the direct sum, taken over all maximal ideals  $\mathfrak{m}$  of  $R$ , of the injective hulls of the residue fields  $R/\mathfrak{m}$ . Fix an element  $\gamma \in R^{\circ, I}$ . We say that  $\gamma$  is an  $(I, *)$ -test element for  $E$  if



for all  $\underline{a}^t = \prod_{i=1}^m \mathfrak{a}_i^{t_i}$ , where the  $\mathfrak{a}_i$  are ideals of  $R$  such that  $\mathfrak{a}_i \cap R^{\circ, I} \neq \emptyset$  and the  $t_i$  are positive real numbers, one has  $\gamma I^{c(q-1)} \mathfrak{a}_1^{\lceil t_1 q \rceil} \dots \mathfrak{a}_m^{\lceil t_m q \rceil} z^q = 0$  in  $\mathbb{F}^e(E)$  for every  $z \in 0_E^{*(I, \underline{a}^t)}$  and for every  $q = p^e$ . If  $R$  is  $F$ -finite and the localized ring  $R_\gamma$  is purely  $F$ -regular along  $IR_\gamma$ , then some power  $\gamma^N$  of  $\gamma$  is an  $(I, *)$ -test element for  $E$ .

*Proof.* It follows from an argument similar to [14] (see also the proofs of [11, Theorem 1.7] and [22, Corollary 3.10 (2)]).  $\square$

**Proposition 2.5.** *Let  $(R, \mathfrak{m})$  be an  $F$ -finite local domain of characteristic  $p > 0$  and  $I$  be an unmixed ideal of height  $c$ . Let  $\underline{a}^t = \prod_{i=1}^m \mathfrak{a}_i^{t_i}$  be a formal combination, where the  $\mathfrak{a}_i$  are ideals of  $R$  such that  $\mathfrak{a}_i \cap R^{\circ, I} \neq \emptyset$  and the  $t_i$  are positive real numbers.*

- (1) *Let  $W$  be a multiplicatively closed subset of  $R$ , and  $\underline{a}_W^t$  and  $I_W$  be the images of  $\underline{a}^t$  and  $I$  in  $R_W$ , respectively. Then*

$$\tilde{\tau}_{I_W}(R_W, \underline{a}_W^t) = \tilde{\tau}_I(R, \underline{a}^t) R_W.$$

- (2) *Let  $\widehat{R}$  be the  $\mathfrak{m}$ -adic completion of  $R$ , and  $\widehat{\underline{a}}^t$  and  $\widehat{I}$  be the images of  $\underline{a}^t$  and  $I$  in  $\widehat{R}$ , respectively. Then*

$$\tilde{\tau}_{\widehat{I}}(\widehat{R}, \widehat{\underline{a}}^t) = \tilde{\tau}_I(R, \underline{a}^t) \widehat{R}.$$

- (3)  *$(R, \underline{a}^t)$  is purely  $F$ -regular along  $I$  if and only if  $\tilde{\tau}_I(R, \underline{a}^t) = R$ .*  
 (4) *If  $R$  is an  $F$ -finite regular local ring and  $\gamma \in R^{\circ, I}$  is an  $(I, *)$ -test element for the injective hull  $E_R(R/\mathfrak{m})$  of the residue field  $R/\mathfrak{m}$ , then  $\tilde{\tau}_I(R, \underline{a}^t)$  is the unique smallest ideal  $J$  of  $R$  with respect to inclusion, such that*

$$\gamma I^{c(q-1)} \mathfrak{a}_1^{\lceil t_1 q \rceil} \dots \mathfrak{a}_m^{\lceil t_m q \rceil} \subseteq J^{[q]}$$

*for all (large)  $q = p^e$ .*

*Proof.* (1) and (2) follow from arguments similar to the proofs of [10, Propositions 3.1 and 3.2], respectively. (3) follows from an argument similar to the proof of [9, Proposition 2.1] (see also [21, Corollary 3.5]) and (4) does from an argument similar to the proof of [2, Proposition 2.22].  $\square$

**Example 2.6.** Let  $(R, \mathfrak{m})$  be an  $F$ -finite regular local ring of characteristic  $p > 0$  and  $I = (f_1, \dots, f_c) \subseteq R$  be an unmixed ideal generated by a regular sequence  $f_1, \dots, f_c$ . Let  $\gamma \in R^{\circ, I}$  be an element such that the localized ring  $R_\gamma/IR_\gamma$  is regular, and take a sufficiently large integer  $N$ . By Definition-Proposition 2.4 and Proposition 2.5 (4),  $R$  is purely  $F$ -regular along  $I$  if and only if there exists  $q = p^e$  such that  $\gamma^N I^{c(q-1)} \not\subseteq \mathfrak{m}^{[q]}$ . Since  $(f_1 \dots f_c)^{q-1} \in I^{c(q-1)} \subseteq (f_1 \dots f_c)^{q-1} R + I^{[q]} = (I^{[q]} : I)$ , this is equivalent to saying that there exists  $q = p^e$  such that  $\gamma^N (I^{[q]} : I) \not\subseteq \mathfrak{m}^{[q]}$ . It therefore follows from [7, Theorem 2.1] that  $R$  is purely  $F$ -regular along  $I$  if and only if  $R/I$  is strongly  $F$ -regular. That is, the generalized test ideal  $\tilde{\tau}_I(R)$  along  $I$  defines the non-strongly- $F$ -regular locus of the ring  $R/I$ .

**Theorem 2.7.** *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional  $F$ -finite regular local ring of characteristic  $p > 0$  and  $I \subseteq R$  be an unmixed ideal of height  $c$ . Let  $\underline{a}^t = \prod_{i=1}^m \mathfrak{a}_i^{t_i}$  be a formal combination, where the  $\mathfrak{a}_i$  are ideals of  $R$  such that  $\mathfrak{a}_i \cap R^{\circ, I} \neq \emptyset$  and the  $t_i$  are positive real numbers. Set  $A = \operatorname{Spec} R$ ,  $X = V(I) \subseteq A$  and  $Y_i = V(\mathfrak{a}_i) \subseteq A$ . Let  $f : A' \rightarrow A$  be the blowing-up of  $A$  along  $X$  and  $E_1, \dots, E_s$  be all the components*

of the exceptional divisor of  $f$  dominating an irreducible component of  $X$ . Suppose that  $\pi : \tilde{A} \rightarrow A$  is a proper birational morphism from a normal scheme  $\tilde{A}$  such that the scheme theoretic inverse images  $\pi^{-1}(X)$  and  $\pi^{-1}(Y_i)$  are Cartier divisors, and denote by  $\tilde{E}$  the strict transform of  $E := E_1 + \cdots + E_S$  on  $\tilde{A}$ . Then one has an inclusion

$$\tilde{\tau}_I(R, \underline{\mathfrak{a}}^t) \subseteq H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - \sum_{i=1}^m \lfloor t_i \pi^{-1}(Y_i) \rfloor - c \pi^{-1}(X) + \tilde{E})).$$

*Proof.* The proof follows from essentially the same argument as that of [11, Proposition 3.8] (see also the proof of [21, Proposition 3.8] for a different strategy). For simplicity, we assume that  $\tilde{A}$  is a Cohen-Macaulay scheme. Denote the closed fiber of  $\pi$  by  $Z := \pi^{-1}(\mathfrak{m})$  and set  $\pi^{-1}(Y) := \sum_{i=1}^m t_i \pi^{-1}(Y_i)$ . Let

$$\delta : H_{\mathfrak{m}}^d(R) \rightarrow H_Z^d(\mathcal{O}_{\tilde{A}}(\lfloor \pi^{-1}(Y) \rfloor + c \pi^{-1}(X) - \tilde{E}))$$

be the edge map  $H_{\mathfrak{m}}^d(R) \rightarrow H_Z^d(\mathcal{O}_{\tilde{A}})$  of the spectral sequence  $H_{\mathfrak{m}}^i(R^j \pi_* \mathcal{O}_{\tilde{A}}) \Rightarrow H_Z^{i+j}(\mathcal{O}_{\tilde{A}})$  followed by the natural map

$$H_Z^d(\mathcal{O}_{\tilde{A}}) \rightarrow H_Z^d(\mathcal{O}_{\tilde{A}}(\lfloor \pi^{-1}(Y) \rfloor + c \pi^{-1}(X) - \tilde{E})).$$

By the local duality theorem (see [12, V, §6]), one has

$$\text{Ann}_R(\text{Ker } \delta) = H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - \lfloor \pi^{-1}(Y) \rfloor - c \pi^{-1}(X) + \tilde{E})).$$

It therefore suffices to show that  $\text{Ker } \delta \subseteq 0_{H_{\mathfrak{m}}^d(R)}^{*(I, \underline{\mathfrak{a}}^t)}$ . Take an element  $\gamma \in R^{\circ, I}$  such that  $R_{\gamma}/IR_{\gamma}$  is regular. Since  $H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c \pi^{-1}(X) + \tilde{E}))_{\gamma} = R_{\gamma}$ , for sufficiently large integers  $N \gg 0$ , one has

$$\begin{aligned} \gamma^N \mathfrak{a}_1^{[t_1]} \cdots \mathfrak{a}_m^{[t_m]} &\subseteq \mathfrak{a}_1^{[t_1]} \cdots \mathfrak{a}_m^{[t_m]} H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c \pi^{-1}(X) + \tilde{E})) \\ &\subseteq H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - \lfloor \pi^{-1}(Y) \rfloor - c \pi^{-1}(X) + \tilde{E})) \\ &= \text{Ann}_R(\text{Ker } \delta). \end{aligned}$$

By Definition-Lemma 2.4, this inclusion tells us that there exists an  $(I, *)$ -test element  $\gamma' \in \text{Ann}_R(\text{Ker } \delta) \cap R^{\circ, I}$  for  $H_{\mathfrak{m}}^d(R)$ , because  $\mathfrak{a}_i \cap R^{\circ, I} \neq \emptyset$  for all  $i = 1, \dots, m$ . For every  $q = p^e$  and for every

$$\alpha \in I^{c(q-1)} \mathfrak{a}_1^{[t_1 q]} \cdots \mathfrak{a}_m^{[t_m q]} \subseteq H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(-q \lfloor \pi^{-1}(Y) \rfloor - c(q-1) \pi^{-1}(X))),$$

we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker } \delta & \longrightarrow & H_{\mathfrak{m}}^d(R) & \xrightarrow{\delta} & H_Z^d(\mathcal{O}_{\tilde{A}}(\lfloor \pi^{-1}(Y) \rfloor + c \pi^{-1}(X) - \tilde{E})) \longrightarrow 0 \\ & & \downarrow & & \downarrow \alpha F^e & & \downarrow \alpha F^e \\ 0 & \longrightarrow & \text{Ker } \delta & \longrightarrow & H_{\mathfrak{m}}^d(R) & \xrightarrow{\delta} & H_Z^d(\mathcal{O}_{\tilde{A}}(\lfloor \pi^{-1}(Y) \rfloor + c \pi^{-1}(X) - \tilde{E})) \longrightarrow 0 \end{array}$$

Then  $\alpha F^e(\text{Ker } \delta) \subseteq \text{Ker } \delta$ . By the choice of the element  $\gamma'$ , we can conclude that  $\gamma' I^{c(q-1)} \mathfrak{a}_1^{[t_1 q]} \cdots \mathfrak{a}_m^{[t_m q]} F^e(\text{Ker } \delta) = 0$  for all  $q = p^e$ , that is,  $\text{Ker } \delta \subseteq 0_{H_{\mathfrak{m}}^d(R)}^{*(I, \underline{\mathfrak{a}}^t)}$ .  $\square$

We conjecture that the generalized test ideal  $\tilde{\tau}_I(R, \underline{\mathfrak{a}}^t)$  along  $I$  corresponds to the adjoint ideal sheaf  $\text{adj}_X(A, Y)$ .

**Conjecture 2.8.** *Let  $(R, \mathfrak{m})$  be a regular local ring essentially of finite type over a perfect field of prime characteristic  $p$ , and let  $I \subseteq R$  be a nonzero unmixed ideal. Let  $\underline{\mathfrak{a}}^t = \prod_{i=1}^m \mathfrak{a}_i^{t_i}$  be a formal combination, where the  $\mathfrak{a}_i$  are ideals of  $R$  such that  $\mathfrak{a}_i \cap R^{\circ, I} \neq \emptyset$  and the  $t_i$  are positive real numbers. Set  $A := \operatorname{Spec} R$ ,  $X := V(I)$  and  $Y := \sum_{i=1}^m t_i V(\mathfrak{a}_i)$ . Assume in addition that  $(R, I, \underline{\mathfrak{a}})$  is reduced from characteristic zero to characteristic  $p \gg 0$ , together with a log resolution  $\pi : \tilde{A} \rightarrow A$  of  $(A, X + Y)$  used to define the adjoint ideal sheaf  $\operatorname{adj}_X(A, Y)$  as in Definition 1.6. Then*

$$\operatorname{adj}_X(A, Y) = \tilde{\tau}_I(R, \underline{\mathfrak{a}}^t).$$

Conjecture 2.8 is true if  $X$  is a divisor on  $A$ .

**Theorem 2.9** ([22, Theorem 5.3]). *Let  $(R, \mathfrak{m})$  be a  $\mathbb{Q}$ -Gorenstein normal local ring essentially of finite type over a perfect field of prime characteristic  $p$ , and let  $f$  be a nonzero element of  $R$ . Let  $\underline{\mathfrak{a}}^t = \prod_{i=1}^m \mathfrak{a}_i^{t_i}$  be a formal combination, where the  $\mathfrak{a}_i$  are ideals of  $R$  such that  $\mathfrak{a}_i \cap R^{\circ, fR} \neq \emptyset$  and the  $t_i$  are positive real numbers. Set  $X = \operatorname{Spec} R$ ,  $D := \operatorname{div}_X(f)$  and  $Y := \sum_{i=1}^m t_i V(\mathfrak{a}_i)$ . Assume in addition that  $(R, f, \underline{\mathfrak{a}})$  is reduced from characteristic zero to characteristic  $p \gg 0$ , together with a log resolution  $\pi : \tilde{X} \rightarrow X$  of  $(X, D + Y)$  used to define the adjoint ideal sheaf  $\operatorname{adj}_D(X, Y)$  as in Definition 1.2. Then*

$$\operatorname{adj}_D(X, Y) = \tilde{\tau}_{fR}(R, \underline{\mathfrak{a}}^t).$$

### 3. RESTRICTION FORMULA OF ADJOINT IDEALS

In this section, we formulate a restriction property of the adjoint ideal sheaf  $\operatorname{adj}_X(A, Y)$  involving the l.c.i. defect ideal sheaf  $\mathcal{D}_X$  of  $X$ .

Let  $A$  be a nonsingular variety over an algebraically closed field  $k$  of characteristic zero and  $X$  be a normal Gorenstein closed subvariety of codimension  $c$  of  $A$ . Kawakita [16] then defined the *l.c.i. defect ideal sheaf*  $\mathcal{D}_X$  of  $X$  as follows. Since the construction is local, we may consider the germ at a closed point  $x \in X$ . We take generically a closed subscheme  $Z$  of  $A$  which contains  $X$  and is locally a complete intersection of codimension  $c$ . By Bertini's theorem,  $Z$  is the scheme-theoretic union of  $X$  and another variety  $C^Z$  of codimension  $c$ . Since  $X$  is Gorenstein, the closed subscheme  $D^Z := C^Z|_X$  of  $X$  is a Cartier divisor (see [24, Lemma 1]). Then the l.c.i. defect ideal sheaf  $\mathcal{D}_X$  of  $X$  is defined by

$$\mathcal{D}_X := \sum_{Z \subset A} \mathcal{O}_X(-D^Z),$$

where  $Z$  runs through all the general locally complete intersection closed subschemes of codimension  $c$  which contain  $X$ . Note that the support of  $\mathcal{D}_X$  coincides with the non-locally complete intersection locus of  $X$ . The reader is referred to [16, Section 2] and [5, Section 9.2] for further properties of l.c.i. defect ideal sheaves.

**Theorem 3.1.** *Let  $A$  be a nonsingular variety over an algebraically closed field  $k$  of characteristic zero and  $Y = \sum_{i=1}^m t_i Y_i$  be a formal combination, where the  $t_i$  are positive real numbers and the  $Y_i$  are proper closed subschemes of  $A$ . If  $X$  is a normal Gorenstein closed subvariety of codimension  $c$  of  $A$  which is not contained in the support of any  $Y_i$ , then*

$$\mathcal{I}(X, V(\mathcal{D}_X) + Y|_X) = \operatorname{adj}_X(A, Y) \mathcal{O}_X,$$

where  $\mathcal{D}_X$  is the l.c.i. defect ideal sheaf of  $X$ .

*Proof.* Since the question is local, we consider the germ at a closed point  $x \in X \cap \bigcap_{i=1}^m Y_i \subset A$ . Let  $\mathcal{I}_X \subseteq \mathcal{O}_A$  be the defining ideal sheaf of  $X$  in  $A$ .

Fix a regular function  $\varphi \in \text{adj}_X(A, Y) \setminus \mathcal{I}_X$ . It then follows from Lemma 1.7 (2) that  $\text{mld}(X_{\text{sing}} \cup \bigcup_{i=1}^m Y_i; A, cX + Y - \text{div}_A(\varphi)) > 0$ . Applying [5, Remark 8.5] (see also the proof of [16, Theorem 1.1]), we have

$$\text{mld}(X_{\text{sing}} \cup \bigcup_{i=1}^m (X \cap Y_i); X, V(\mathcal{D}_X) + Y|_X - \text{div}_X(\overline{\varphi})) > 0,$$

where  $\overline{\varphi}$  is the image of  $\varphi$  in  $\mathcal{O}_X$ . This means that  $(X, V(\mathcal{D}_X) + Y|_X - \text{div}_X(\overline{\varphi}))$  is klt, which is equivalent to saying that  $\overline{\varphi}$  is in  $\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X)$ . Thus, we conclude that

$$\text{adj}_X(A, Y)\mathcal{O}_X \subseteq \mathcal{J}(X, V(\mathcal{D}_X) + Y|_X).$$

Next we will prove the converse inclusion. Take generically a closed subscheme  $Z$  of  $A$  which contains  $X$  and is locally a complete intersection of codimension  $c$ , so  $Z$  is the scheme-theoretic union of  $X$  and another variety  $C^Z$ , and  $D^Z := C^Z|_X$  is a Cartier divisor on  $X$ .

*Claim 1.* By a general choice of  $Z$ , one has

$$\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X) = \text{adj}_{D^Z}(X, Y|_X).$$

*Proof of Claim 1 (Kawakita).* Since  $\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X) \supseteq \text{adj}_{D^Z}(X, Y|_X)$  is clear from the definition of the ideal sheaf  $\mathcal{D}_X$ , we will prove the converse inclusion.

Fix a regular function  $\psi \in \mathcal{J}(X, V(\mathcal{D}_X) + Y|_X)$ . By the definition of the ideal sheaf  $\mathcal{D}_X$ , there exist closed subschemes  $W_1, \dots, W_n$  of  $A$  which contain  $X$  and are locally complete intersections of codimension  $c$  such that  $\mathcal{D}_X = \sum_{j=1}^n \mathcal{O}_X(-D^{W_j})$ . Take a log resolution  $\mu : \tilde{X} \rightarrow X$  of  $(X, D^{W_1} + \dots + D^{W_n} + Y|_X)$ , and let  $\{E_i\}_{i \in I}$  be a collection of all divisors on  $\tilde{X}$  which are supported on  $\text{Exc}(\mu) \cup \bigcup_{i=1}^m \text{Supp } \mu^{-1}(Y_i|_X) \cup \bigcup_{j=1}^n \mu_*^{-1} D^{W_j}$ . Since  $\psi$  is in  $\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X)$ , we have

$$\text{ord}_{E_i}(\psi) + \max_{1 \leq j \leq n} a(E_i; X, D^{W_j} + Y|_X) > 0$$

for all  $i \in I$ . Let  $I_{W_j} = (f_1^{(j)}, \dots, f_c^{(j)}) \subseteq \mathcal{O}_A$  be the defining ideal sheaf of  $W_j$  in  $A$ , and set  $I_{\mathcal{W}} := (t_1 f_1^{(1)} + \dots + t_n f_1^{(n)}, \dots, t_1 f_c^{(1)} + \dots + t_n f_c^{(n)}) \subseteq \mathcal{O}_A[t_1, \dots, t_n]$ . Let  $\mathcal{W} \subseteq A \times T$  be the corresponding closed subscheme, where  $T := \text{Spec } k[t_1, \dots, t_n]$ . Then  $\mathcal{W}$  is the scheme-theoretic union of  $X \times T$  and another variety  $\mathcal{C}$ . Note that  $\mathcal{C}|_{X \times T}$  is an irreducible Cartier divisor over a generic point of  $T$ . One can choose a generator  $h \in \mathcal{O}_X \otimes k(t_1, \dots, t_n)$  of the principal ideal sheaf  $\mathcal{O}_{X \times T}(-\mathcal{C}|_{X \times T})$  over a generic point of  $T$  so that the restriction of  $h$  to the fiber over  $(0, \dots, 0, \overset{j}{1}, 0, \dots, 0) \in T$  is a generator of  $\mathcal{O}_X(-D^{W_j})$ . Thinking of the resolution  $\mu \times \text{id}_T : \tilde{X} \times T \rightarrow X \times T$  induced by  $\mu$ , we have

$$a(\mathcal{E}_i; X \times T, \text{div}(h) + Y|_{X \times T}) \geq a(E_i; X, D^{W_j} + Y|_X)$$

for all  $i \in I$  and all  $j = 1, \dots, n$ , where  $\mathcal{E}_i := E_i \times T \subset \tilde{X} \times T$ .

On the other hand, the restriction of  $h$  to the fiber over a general point  $(t_1, \dots, t_n) \in T$  is a generator of  $\mathcal{O}_X(-D^{W_{t_1, \dots, t_n}})$  where  $W_{t_1, \dots, t_n} := \mathcal{W}|_{X \times (t_1, \dots, t_n)}$ , and  $\mu$  is a log resolution of  $(X, W_{t_1, \dots, t_n} + Y|_X)$ . Thus, for all  $i \in I$ ,

$$\begin{aligned} \text{ord}_{E_i}(\psi) + a(E_i; X, D^{W_{t_1, \dots, t_n}} + Y|_X) &= \text{ord}_{E_i}(\psi) + a(\mathcal{E}_i; X \times T, \text{div}(h) + Y|_{X \times T}) \\ &\geq \text{ord}_{E_i}(\psi) + \max_{1 \leq j \leq n} a(E_i; X, D^{W_j} + Y|_X) \\ &> 0. \end{aligned}$$

This implies that  $\psi$  is in  $\text{adj}_{D^{W_{t_1, \dots, t_n}}}(X, Y|_X)$ .  $\square$

From now on, we may assume that  $A = \text{Spec } S$  and  $X = \text{Spec } R$ , where  $(S, \mathfrak{n})$  is a regular local ring essentially of finite type over a field of characteristic zero and  $R = S/I$  is a Gorenstein normal quotient of  $S$ . Let  $\mathfrak{a}_i$  be the ideal of  $S$  corresponding to  $Y_i$  for every  $i = 1, \dots, n$  and denote  $\mathfrak{a}^t = \prod \mathfrak{a}_i^{t_i}$ . Let  $f_1, \dots, f_c$  be the regular sequence in  $S$  corresponding to  $Z$  and  $f \in S$  be an element whose image  $\bar{f}$  is a generator of the principal ideal  $((f_1, \dots, f_c) : I) + I/I$  of  $R$ . Thanks to Claim 1, it is enough to prove that

$$(1) \quad \text{adj}_{\text{div}(\bar{f})}(X, Y|_X) \subseteq \text{adj}_X(A, Y)\mathcal{O}_X.$$

Now we reduce the entire setup as above to characteristic  $p \gg 0$  and switch the notation to denote things after reduction modulo  $p$ . In order to prove (1), by virtue of Theorems 2.7 and 2.9, it suffices to show that

$$(2) \quad \tilde{\tau}_{fR}(R, \mathfrak{a}R^t) \subseteq \tilde{\tau}_I(S, \mathfrak{a}^t)R.$$

Since  $S$  is F-finite, by Lemma 2.5, forming generalized test ideals commutes with completion. Hence, we may assume that  $S$  is complete. Let  $E_S = E_S(S/\mathfrak{n})$  and  $E_R = E_R(R/\mathfrak{n}R)$  be the injective hulls of the residue fields of  $S$  and  $R$ , respectively. We can view  $E_R$  as a submodule of  $E_S$  via the isomorphism  $E_R \cong (0 : I)_{E_S} \subset E_S$ . Then by Matlis duality, (2) is equivalent to saying that

$$(3) \quad 0_{E_R}^{*(fR, \mathfrak{a}R^t)} \supseteq 0_{E_S}^{*(I, \mathfrak{a}^t)} \cap E_R.$$

Let  $z \in 0_{E_S}^{*(I, \mathfrak{a}^t)} \cap E_R$ . Let  $F_S^e : E_S \rightarrow \mathbb{F}_S^e(E_S) \cong E_S$  and  $F_R^e : E_R \rightarrow \mathbb{F}_R^e(E_R) \cong E_R$  be the  $e$ -times iterated Frobenius maps induced on  $E_S$  and  $E_R$ , respectively. Since  $R = S/I$  is normal, one can choose an element  $\gamma \in S^{\circ, I}$  such that the image  $\bar{\gamma}$  of  $\gamma$  is not contained in any minimal prime of  $fR$  and the localized ring  $R_{\bar{\gamma}}$  is regular. By Definition-Lemma 2.4, some power  $\gamma^N$  of  $\gamma$  is an  $(I, *)$ -test element for  $E_S$ , and then  $\gamma^N I^{c(q-1)} \mathfrak{a}_1^{[t_1 q]} \dots \mathfrak{a}_m^{[t_m q]} F_S^e(z) = 0$  for all  $q = p^e$ . On the other hand,  $I^{[q]} F_S^e(z) = 0$  for all  $q = p^e$ , because  $z \in E_R \cong (0 : I)_{E_S}$ .

*Claim 2.* For all  $q = p^e$ , one has

$$f^{q-1}(I^{[q]} : I) \subseteq I^{c(q-1)} + I^{[q]}.$$

*Proof of Claim 2.* This claim is an easy consequence of the linkage theory of Peskine and Szpiro [19]. We consider a simultaneous minimal free resolution of a natural

diagram between  $S/(f_1, \dots, f_c), S/I, S(f_1^q, \dots, f_c^q)$  and  $S/I^{[q]}$ .

$$\begin{array}{ccccccc}
0 & \longrightarrow & S & \longrightarrow & \cdots & \longrightarrow & S/(f_1^q, \dots, f_c^q) \longrightarrow 0 \\
& & \downarrow \times (f_1 \cdots f_c)^{q-1} & & & & \downarrow \\
0 & \longrightarrow & S & \longrightarrow & \cdots & \longrightarrow & S/(f_1, \dots, f_c) \longrightarrow 0 \\
& & \downarrow \times f & & & & \downarrow \\
0 & \longrightarrow & S & \longrightarrow & \cdots & \longrightarrow & S/I^{[q]} \longrightarrow 0 \\
& & \downarrow & & & & \downarrow \\
0 & \longrightarrow & S & \longrightarrow & \cdots & \longrightarrow & S/I \longrightarrow 0
\end{array}$$

Here note that the ideals  $I, I^{[q]}, (f_1, \dots, f_c)$  and  $(f_1^q, \dots, f_c^q)$  are all Gorenstein of height  $c$ . Looking at the last step of the above diagram, by [24, Lemma 1], we obtain the equality  $f^q(I^{[q]} : I) = f(f_1 \cdots f_c)^{q-1}$  in  $S/I^{[q]}$ . Since  $f$  is a regular element of  $S/I$ , by the flatness of the Frobenius map,  $f$  is also a regular element of  $S/I^{[q]}$ . Therefore,  $f^{q-1}(I^{[q]} : I) = (f_1 \cdots f_c)^{q-1}$  in  $S/I^{[q]}$ , which gives the assertion.  $\square$

Thanks to Claim 2, we have  $\gamma^N f^{q-1}(I^{[q]} : I) \mathbf{a}_1^{[t_1 q]} \cdots \mathbf{a}_m^{[t_m q]} F_S^e(z) = 0$  for all  $q = p^e$ . This is equivalent to saying that  $\bar{\gamma}^N \bar{f}^{q-1} (\mathbf{a}_1 R)^{[t_1 q]} \cdots (\mathbf{a}_m R)^{[t_m q]} F_R^e(z) = 0$  for all  $q = p^e$ , because  $\mathbb{F}_R^e(E_R) \cong (0 : I^{[q]})_{E_S} / (0 : (I^{[q]} : I))_{E_S}$  (see [6], [7] and the proof of [21, Lemma 3.9]). Since  $\bar{\gamma}$  is not in any minimal prime of  $fR$ , we conclude that  $z \in 0_{E_R}^{*(fR, \underline{a}R^{\pm})}$ .  $\square$

*Remark 3.2.* (1) In the proof of Theorem 3.1, we have used [5, Remark 8.5] which was originally proved by using the theory of jet schemes. It, however, can be proved without using the theory of jet schemes (see the proof of [16, Theorem 1.1]). So, our proof does not rely on the theory of jet schemes.

(2) In the statement of [5, Theorem 1.1], the coefficients of  $Y$  have to be nonnegative. Therefore, Theorem 3.1 does not follow from their result.

(3) The l.c.i. defect ideal sheaf can be defined even when  $X$  is only  $\mathbb{Q}$ -Gorenstein. Even in this case, Kawakita [16] and Ein-Mustařă [5] formulated a comparison of minimal log discrepancies of  $X$  and  $A$ . We, therefore, expect a generalization of Theorem 3.1 to the  $\mathbb{Q}$ -Gorenstein case, but it is open.

**Example 3.3.** (1) Let  $X = \frac{1}{3}(1, 1, 1)$  be the quotient of  $\mathbb{C}^3 = \text{Spec } \mathbb{C}[x_1, x_2, x_3]$  by the action of  $\mathbb{Z}/3\mathbb{Z}$  given by  $x_i \mapsto \xi x_i$ , where  $\xi$  is a primitive cubic root of unity. Denote by  $\mathfrak{m}_{X,0} \subset \mathcal{O}_X$  (resp.  $\mathfrak{m}_{\mathbb{C}^3,0} \subset \mathcal{O}_{\mathbb{C}^3}$ ) the maximal ideal sheaf of the origin. By [16, Example 2.3], the integral closure of the l.c.i. defect ideal sheaf  $\mathcal{D}_X$  of  $X$  is  $\mathfrak{m}_{X,0}^5$ . Therefore,

$$\begin{aligned}
\mathcal{J}(X, V(\mathcal{D}_X)) &= \mathcal{J}(X, \mathfrak{m}_{X,0}^5) = \mathcal{J}(\mathbb{C}^3, \mathfrak{m}_{\mathbb{C}^3,0}^{15}) \cap \mathcal{O}_X \\
&= \mathfrak{m}_{\mathbb{C}^3,0}^{13} \cap \mathcal{O}_X \\
&= \mathfrak{m}_{X,0}^5.
\end{aligned}$$

On the other hand,  $X$  can be embedded into  $A := \mathbb{C}^{10}$ . Then we have already seen in Example 1.8 (2) that  $\text{adj}_X(A) = \mathfrak{m}_{A,0}^5$ , where  $\mathfrak{m}_{A,0} \subset \mathcal{O}_A$  is the maximal ideal sheaf of the origin. Thus,  $\mathcal{J}(X, V(\mathcal{D}_X)) = \text{adj}_X(A)\mathcal{O}_X = \mathfrak{m}_{X,0}^5$ .

(2) Let  $X$  be the quotient of  $(x^2 + y^3 + z^6 = 0) \subset \mathbb{C}^3$  by the action of  $\mathbb{Z}/5\mathbb{Z}$  given by  $x \mapsto \xi^3 x$ ,  $y \mapsto \xi^2 y$  and  $z \mapsto \xi z$ , where  $\xi$  is a primitive quintic root of unity. Then  $X$  can be embedded into  $A := \mathbb{C}^5$ . Since  $X$  is a Gorenstein closed subvariety of codimension three of  $A$ , by [16, Example 2.4] (which is an application of the structure theorem for Gorenstein ideals of codimension three in [1]), the l.c.i. defect ideal sheaf  $\mathcal{D}_X$  of  $X$  is the maximal ideal sheaf  $\mathfrak{m}_{X,0}$  of the origin. Let  $\pi : \tilde{X} \rightarrow X$  be the blowing-up of  $X$  at the origin with exceptional divisor  $E$ . Then  $\pi$  is a log resolution of  $X$  and  $K_{\tilde{X}/X} = -E$ . Therefore,

$$\mathcal{J}(X, V(\mathcal{D}_X)) = \pi_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/X} - E) = \pi_* \mathcal{O}_{\tilde{X}}(-2E) = \mathfrak{m}_{X,0}^2.$$

On the other hand, we have already seen in Example 1.8 (3) that  $\text{adj}_X(A) = \mathfrak{m}_{A,0}^2$ , where  $\mathfrak{m}_{A,0} \subset \mathcal{O}_A$  is the maximal ideal sheaf of the origin. Thus,  $\mathcal{J}(X, V(\mathcal{D}_X)) = \text{adj}_X(A)\mathcal{O}_X = \mathfrak{m}_{X,0}^2$ .

We conclude this section by stating a corollary of Theorem 3.1.

Let  $S$  be an algebra essentially of finite type over a field  $k$  of characteristic zero, and let  $I \subseteq S$  be an unmixed ideal of height  $c$ . Let  $\underline{\mathfrak{a}}^t = \prod_{i=1}^m \mathfrak{a}_i^{t_i}$  be a formal combination, where the  $\mathfrak{a}_i$  are ideals of  $S$  such that  $\mathfrak{a}_i \cap S^{\circ, I} \neq \emptyset$  and the  $t_i$  are positive real numbers. We say that  $(S, \underline{\mathfrak{a}}^t)$  is of *purely F-regular type* along  $I$  if there exist a finitely generated  $\mathbb{Z}$ -subalgebra  $A$  of  $k$  and an algebra  $S_A$  essentially of finite type over  $A$  satisfying the following conditions:

- (i)  $S_A$  is flat over  $A$ . In addition,  $S_A \otimes_A k \cong S$ ,  $I_A S = I$  and  $\underline{\mathfrak{a}}_A S = \underline{\mathfrak{a}}$ , where  $I_A := I \cap S_A \subseteq S_A$  and  $\underline{\mathfrak{a}}_A$  is the restriction of  $\underline{\mathfrak{a}}$  to  $S_A$ .
- (ii)  $(S_\kappa, \underline{\mathfrak{a}}_\kappa^t)$  is purely F-regular along  $I_\kappa$  for every closed point  $s$  in a dense open subset of  $\text{Spec } A$ , where  $\kappa = \kappa(s)$  denotes the residue field of  $s \in \text{Spec } A$ ,  $S_\kappa = S_A \otimes_A \kappa(s)$ ,  $I_\kappa := I_A S_\kappa$  and  $\underline{\mathfrak{a}}_\kappa$  is the image of  $\underline{\mathfrak{a}}_A$  in  $S_\kappa$ .

**Corollary 3.4.** *In the above situation, suppose in addition that  $k$  is an algebraically closed field,  $S$  is a regular domain and  $S/I$  is a Gorenstein quotient of  $S$ . Then  $(\text{Spec } S, \sum_{i=1}^m t_i V(\mathfrak{a}_i))$  is plt along  $V(I)$  if and only if  $(S, \underline{\mathfrak{a}}^t)$  is of purely F-regular type along  $I$ .*

*Proof.* Since the statement is local, we may assume that  $S$  is a regular local ring. The “if” part immediately follows from Proposition 2.5 (3) and Theorem 2.7. We will prove the “only if” part.

Suppose that  $(\text{Spec } S, \sum_{i=1}^m t_i V(\mathfrak{a}_i))$  is plt along  $V(I)$ . Then  $R := S/I$  is a normal local ring. Let  $f_1, \dots, f_c$  be a general regular sequence in  $S$  and  $f \in S$  be an element whose image  $\bar{f}$  is a generator of the principal ideal  $((f_1, \dots, f_c) : I) / I$  of  $R$ . Then, by Theorem 3.1 and Claim 1 in the proof of Theorem 3.1, the pair  $(\text{Spec } R, \sum_{i=1}^m t_i V(\mathfrak{a}_i R))$  is plt along  $\text{div}(\bar{f})$ . Thanks to Theorem 2.9, this implies that  $(R, \underline{\mathfrak{a}} R^t)$  is of purely F-regular type along  $fR$ . Applying the argument used to prove the inclusion (2) in the proof of Theorem 3.1, we know that  $(S, \underline{\mathfrak{a}}^t)$  is of purely F-regular type along  $I$ .  $\square$

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